

Stagnation, Creation, Breaking

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Abstract

This note deals with the mono-dimensional equation: $\partial_t u - \partial_x L(u_x) = f$ with $L(\cdot)$ merely monotone. The goal is to examine the features of facets – flat regions of graphs of solutions appearing as $L(\cdot)$ suffers jumps. We concentrate on examples explaining strong stability, creation and even breaking of facets.

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1 Introduction

Facets, flat regions of interfaces are inseparable elements of phenomena of a large class of anisotropic phase transition models [9, 32, 33]. They are natural in the crystal growth theory, but also in other phenomena where isotropy is broken by external conditions [3, 12, 18, 23, 34]. Creation of such regions is explained at the microscopic level. The anisotropy distinguishes selected directions, and then the energy related to the interface is minimized over such structure. However in real processes the structure of interface may be very rough, using a common language, it is hard to call it *smooth*. Effects of creation and breaking are typically present, in addition one can observe very stable (local) structures.

Mathematical modeling of such phenomena is still far from perfect. We have some systems formal derived from classical theories. Such systems are strongly nonlinear, the nonlinearities are so high they cause nonlocal effects [8, 11, 22, 31]. On other hand the current theory of PDEs is not able to handle all mathematical challenges. Thus, it is a perfect area to attempt new mathematical theories. The first step is to understand key phenomena from the viewpoint of mathematical analysis in order to build new sets of analytical techniques. And this note is dedicated for this purpose.

In our considerations we address to mono-dimensional equations. Obstacles appearing in multidimensional problems are related the most with geometry. It causes very high complexity. The 1d case can be viewed as a toy model, but we shall remember that it can be adapted for description of two dimensional phenomena [12, 17, 35]. Here we study the following class of parabolic equations

$$u_t - \partial_x L(u_x) = f \quad (1.1)$$

on an interval with suitable initial and boundary conditions. Operator $L(\cdot)$ is required to be monotone. It is well known that jumps (points of discontinuity) of $L(\cdot)$ cause unusual structure of graphs of solutions. Namely, it generates flat regions, called by the theory: *facets* – [13, 22]. The key element is the r.h.s. of (1.1) – the external force f . The mathematical goal of this paper is to look closer at structure of solutions to (1.1) and understand interaction between f and the facets.

The aim of the note is to illustrate three phenomena: stagnation, creation, breaking of flat regions of solutions of (1.1). In the mathematical language our analysis will concern:

- ♣ stagnation, strong stability of solutions, a change of data does not change the solution;
- ♣ creations of facets, we prove that from the mathematical point of view such shapes are typical, even locally;
- ♣ breaking the facet, this phenomenon is spectacular, it is always associated with a large external force, it can not be a consequence of initial state.

The stagnation has been noted in results of [24, 26]. The creation was examined in [21, 29]. The phenomenon of breaking was a subject of many results. We shall mention here the results from [6, 7] concerning the weighted curvature flow in the three dimensional case, but without the force. The case of bending/breaking facets under influence of the external force has been investigated in [14, 15, 16].

Mathematical problems stated above are arising not only from modeling of evolution of crystals. Such equations arise also from the image processing [10, 19, 24, 25]. We shall keep in mind that the subject can be viewed as a limit case of many general mathematical investigations of nonlinear parabolic systems [1, 2, 4, 5, 20], too. In other words we will look at mathematical results as interesting on their own. Aspects of applications here are secondary.

This note is a successive step for analysis of (1.1)-type systems. The key difference between the problem from [21, 27, 29, 30] and this note, is presence of the external force. Thus, here we continue our investigations from [26], where the stationary problem has been examined.

2 The issue

Let us describe the problem and basic assumptions. We will examine the following class of equations

$$u_t - \partial_x L(u_x) = f \text{ in } (0, 1) \times (0, T) \quad (2.1)$$

with the Dirichlet boundary data, and suitable initial state

$$u(0, t) = u(1, t) = 0 \quad \text{for } t \in [0, T] \quad \text{and} \quad u|_{t=0} = u_0 \text{ in } (0, 1). \quad (2.2)$$

The initial datum will be suitably chosen, and the external force f will be sufficiently smooth. We shall keep in mind we are interested in description of typical of solutions, thus the full generality of data is not considered in this note. General form of $L(\cdot)$ does not allow to pinpoint the most interesting examples of behavior of solutions. From that reason we distinguish two cases:

$$L_0(p) = \text{sgn } p \quad \text{and} \quad L_1(p) = p + \text{sgn } p. \quad (2.3)$$

Nevertheless all operators prescribed by $L(\cdot)$ suffering jumps will be in the scope of main interest. Due to required simplicity we concentrate on the case with just one jump like for $L_1(\cdot)$.

Let us list main results of our investigations:

◆ First, we discuss regularity of solutions. Based on the results from [28] we show the existence of solutions. In addition we are allowed to control the L_∞ bound of u_t as well as $L(u_x)_x$, provided suitable information about the initial state.

◆ Section 4 is dedicated to strong stability – stagnation – of facets. In this part we identify a class of initial data, which are insensitive to changing of the external force. These examples suggest a very interesting hypothesis/question: how the structure of the external force determines the changes of solutions.

◆ The next result explains the effect of creation of facets. We prove that, except initial time, all extrema must be realized on nontrivial intervals.

◆ The last point of our investigations is analysis of the phenomenon on breaking facets. We show it just for one concrete example, but these considerations describe a generic case for arbitrary flat regions.

Throughout the paper we are trying to keep the standard notation.

3 Existence, uniqueness, regularity

We start with the basic existence result. The monotonicity of the operator $\partial_x L(\partial_x \cdot)$ immediately yields the uniqueness of solutions to (2.1)-(2.2). In [28, 29] we find the following result specifying regularity of unique solutions to systems of (2.1)-(2.2) type.

Theorem 3.1 *Let $L'(\cdot) \geq 0$, $u_0 \in L_1(I)$, $u_{0,x} \in BV(I)$, $u_0|_{\{0,1\}} = 0$, $f \in C([0, T]; W_\infty^1(I))$. Then there exists a unique weak solution solution to (2.1)-(2.2) such that*

$$u_x \in L_\infty(0, T; BV(I)), \quad u_t \in L_\infty(\delta, T; L_2(I)) \text{ for } \delta > 0. \quad (3.1)$$

If in addition $L'(\cdot) \geq d > 0$ for a constant d , then $u_{xx} \in L_\infty(\delta, T; L_2(I))$.

Very important information is regularity of studied solutions. We prove that for merely monotone $L(\cdot)$ the time derivative u_t is point-wise bounded in the meaning of the L_∞ space, provided that initial value of u_t is defined suitably.

Lemma 1 *Let $L'(\cdot) \geq 0$. Let $\partial_t f \in L_1(0, T; L_\infty(I))$, $u_t|_{t=0} \in L_\infty(I)$, then*

$$\sup_{t \in [0, T]} \|u_t\|_{L_\infty(I)} \leq C(\|u_t|_{t=0}\|_{L_\infty(I)} + \|f_t\|_{L_1(0, T; L_\infty(I))}). \quad (3.2)$$

Proof. We put our attention on formal estimates, purely analytical investigations shall be done at the level of the approximation, see [28]. Differentiate the equation with respect to t

$$u_{tt} - \partial_x(L'(u_x)u_{tx}) = f_t. \quad (3.3)$$

Note that the differentiation in time does not change the boundary conditions. The simplest approach is by the Moser technique. We test the solution by $|u_t|^{p-1}u_t$ getting $\frac{d}{dt}\|u_t\|_{L_p(I)} \leq \|f_t\|_{L_\infty(I)}$. In the r.h.s. should appear a constant depending on the measure of set I , but in our case it is just 1. Eventually, we prove

$$\sup_{t \in [0, T]} \|u_t(\cdot, t)\|_{L_p(I)} \leq \|u_0|_{t=0}\|_{L_\infty} + \int_0^T \|f_t\|_{L_\infty(I)}. \quad (3.4)$$

Passing with $p \rightarrow \infty$ we get (3.2). The Lemma is proved. \square

Here we shall put a question concerning the initial data for u_t , this value can be obtained only from the equation, it means we have to clarify the meaning of

$$\partial_x L(u_{0,x}).$$

This difficulty is related to the programme realized by the author and coworkers [21, 29], however still the answer for general system is highly non-predictable. The theory [21] explains this case

just for the mono-dimensional total variation flow, namely for $L_0(\cdot)$. Hence here we point two cases which describe us reasonable classes when $u_t|_{t=0}$ is determined.

The first one is for $u_t|_{t=0} = 0$, this case holds as u_0 is a stationary solution to the following equation

$$\partial_x L(u_{0,x}) = f(0). \quad (3.5)$$

In other words, we have

Lemma 2 *If u_0 fulfills the equation (3.5), then if we consider the evolutionary system (2.1) with the initial data u_0 and f such that $f(0)$ is the r.h.s. of (3.5), then $u_t|_{t=0} = 0$.*

The proof follows from the definition and results of [26]. Thanks to the uniqueness we are able to study the case of an approximation of the system such that $f(s) = f(0)$ for $s \in [0, \epsilon]$. Then we are ensured that $u \equiv u_0$ on the interval $[0, \epsilon]$. Thus, its time derivative is zero as well. In particular it is zero for $t = 0$ for all ϵ , passing with ϵ to zero, we obtain the thesis of Lemma in the general form. \square

Any general answer of the question of the meaning of $u_t|_{t=0}$ is related to definition of the initial quantity: $\partial_x L(u_x) - f$. From the considerations from [26] we still are not able to determine it in a useful way. However we are allowed to consider a very special case for $f(0) = 0$. Then we meet results from [21], for the total variational flow in the mono-dimensional case. They say that if the initial data belong to an admissible class (which is dense in a suitable class of regularity) then $u_t|_{t=0} \in L_\infty$. So the case of L_0 is done. In general the situation is more complex, the know result describes the idea of the almost classical solutions just to convex solutions [28]. We do not consider this case since we expect better explanation of this issue in the forthcoming results. A need of redefining the meaning of almost classical solutions to requirements of inhomogeneous systems is required.

4 Stagnation

The first feature, we would like to investigate, is the stagnation of facets. The observation says the length of the flat region depends of the total force acting on this part. In other words if the change of the force won't change the average then we do not expect change of the length of the facet. Our examples show even that there will be no influence on the rest of the solution. Thus we obtain the stagnation effect for the whole solution.

To illustrate this phenomenon we consider the case $L_1(\cdot)$. The analysis can be generalized on the following case

$$L_2(p) = \text{sgn } p + L_r(p), \quad (4.1)$$

where L_r is a regular C^1 function. In other words $L_2(p)$ may suffer just one jump for $p = 0$.

Let us first consider the simplest case. We prove

Lemma 3 *Consider $L(\cdot) = L_2(\cdot)$ defined by (4.1). Let $u_0 \equiv 0$, f be a smooth function. Then*

$$u \equiv 0 \text{ on } I \times (0, T), \quad (4.2)$$

provided

$$\int_0^x f(x', t) dx' \in [-1 + c, 1 + c] \quad \text{for } t \in [0, t) \quad \text{for some constant } c. \quad (4.3)$$

Proof. We construct the solution. Put

$$\sigma(x, t) = -c + \int_0^x f(x', t) dx' \in [-1, 1] = L_2(0). \quad (4.4)$$

Then $u \equiv 0$, since it fulfills the weak formulation of the problem, which is

$$(0, \phi) + (\sigma, \phi_x) = (f, \phi) \text{ for } \phi \in C_0^\infty(0, 1). \quad (4.5)$$

□

Remark. To clarify (4.4) we recall the definition of weak solutions to (2.1). Let $u \in L_\infty(0, T; L_2(\Omega))$ and $\sigma \in L_1(0, T; L_1(I))$ such that

$$\sigma(x, t) \in L(u_x(x, t)) \text{ a.e.,}$$

where $L(u_x)$ is just the composition of multivalued operators; and the following identity holds

$$(u_t, \phi) + (\sigma, \phi_x) = (f, \phi) \text{ in } \mathcal{D}'([0, T]) \quad (4.6)$$

for all $\phi \in C^\infty([0, T]; C_0^\infty(I))$. The u is a weak solution to (2.1). Such solutions are unique.

Next, we point a more complex case for L_1 .

Lemma 4 *Consider $L(\cdot) = L_1(\cdot)$ defined in (2.3). Let $f \in C([0, T]; L_\infty(I))$, u_0 be a steady solution to the problem*

$$-\partial_x(u_x + \operatorname{sgn} u_x) = f(0) \quad (4.7)$$

and $f(0) \geq 0$. In addition we require that $f(0)$ is chosen in such a way that there exist points $0 < \xi_- < \xi_+ < 1$ such that

$$\int_{\xi_-}^{\xi_+} f(s, 0) ds = 2. \quad (4.8)$$

Then the solution u to the following problem

$$u_t - \partial_x(u_x + \operatorname{sgn} u_x) = f(t) \quad (4.9)$$

is static, it is

$$u(x, t) = u_0(x) \text{ for } (x, t) \in I \times [0, T], \quad (4.10)$$

provided $f(\cdot)$ fulfills:

- ♠ *$\operatorname{supp} f(t) - f(0) \subset [\xi_-, \xi_+]$ for $t \in [0, T]$;*
- ♠ *$\int_{\xi_-}^{\xi_+} f(x, t) dx = 2$;*
- ♠ *$|\int_a^b f(x, t) dx| < 2$ for all a, b such that $\xi_- < a < b < \xi_+$.*

Proof. The proof follows from considerations in [26], we show that for all t the force $f(t)$ generates the same steady solution, so (4.9) must hold. A key idea follows from regularity of the solutions. Theorem 1 from [26] implies the Hölder inequality of u_x , so we split our analysis into a monotone part, and there we analyze the heat equation, and at the area where the solution is flat. So a natural boundary condition at the interface is $u_x = 0$. In the studied case the solution is expected to be time independent, so it is enough to check whether it fulfills the weak formulation, and then the uniqueness ends our considerations. We keep in mind that all solutions here are unique.

In Section 6 we prove a more complex result. Lemma 4 is viewed as a particular case of considerations for Lemmas 6 and 7.

□

5 Creation

Next, we consider the phenomenon of creation of flat region of solution. A result of our analysis will be the rule that if $L(\cdot)$ has a jump for value \bar{p} , then the set $\{u_x = \bar{p}\}$ consists of isolated non-degenerated closed intervals. We illustrate this behavior concentrated on the model case $L_2(\cdot)$. Thus we study only L with jump for $\bar{p} = 0$. In particular, the presence of the force can not break this rule.

Lemma 5 *Let $L'(\cdot) \geq 0$ and f be a smooth function. Then each local extremum of the solution $u(\cdot, t)$ is realized over non-degenerated interval for $t > 0$.*

Proof. For fixed $t \in [0, T]$ the space derivative $u_x(\cdot, t) \in TV(I)$, hence $u \in W_\infty^1(I)$. Assume that we consider a maximum at point $x_0 \in I$. Then since u_x is a TV function, one can find a sequence $x_n^-(t)$ and $x_n^+(t)$ such that

$$x_n^-(t) \rightarrow x_0^- \quad \text{and} \quad x_n^+(t) \rightarrow x_0^+, \quad (5.1)$$

such that

$$u(x_n^-, t), u(x_n^+, t) < u(x_0, t) \quad \text{and} \quad u_x(x_n^-, t) > 0, \quad u_x(x_n^+, t) < 0. \quad (5.2)$$

The integration of the equation (2.1) over the interval $[x_n^-(t), x_n^+(t)] \times \{t\}$ yields

$$-L(u_x)|_{(x_n^-(t), t)}^{(x_n^+(t), t)} = \int_{x_n^-(t)}^{x_n^+(t)} (f(y, t) - u_t(y, t)) dy. \quad (5.3)$$

The choice of sequences implies

$$u_x(x_n^-(t), t) > 0 \quad \text{and} \quad u_x(x_n^+(t), t) < 0. \quad (5.4)$$

Hence we obtain at the limit the following inequality

$$2 \leq (x_0^+ - x_0^-)^{1/2} \left(\int_{x_0^-}^{x_0^+} |f|^2 + |u_t|^2 dx \right)^{1/2} \leq (x_0^+ - x_0^-)^{1/2} (\|f\|_{L_\infty}(t) + \|u_t\|_{L_2}(t)). \quad (5.5)$$

To justify (5.5) we note that since (5.2) holds then

$$\liminf_{n \rightarrow \infty} L(u_x(x_n^+, t)) \geq 1 \quad \text{and} \quad \limsup_{n \rightarrow \infty} L(u_x(x_n^-, t)) \leq -1. \quad (5.6)$$

It follows that $x_0^- < x_0^+$ and the maximum must hold on a nontrivial interval. On the other hand we obtain the inequality on the lower bound of the length of the facet. From (5.5) we get

$$x_0^+(t) - x_0^-(t) \geq \left(\frac{2}{\|f\|_{L_\infty}(t) + \|u_t\|_{L_2}(t)} \right)^2. \quad (5.7)$$

By Theorem 3.1, the denominator is finite for $t > 0$, so $x_0^- \neq x_0^+$. The lower bound (5.7) can be improved provided we have better information about u_t . In case of Lemma 1 we find then the following bound

$$x_0^+(t) - x_0^-(t) \geq \left(\frac{2}{\|f\|_{L_\infty}(t) + \|u_t\|_{L_\infty}(t)} \right)^2. \quad (5.8)$$

□

6 Breaking

In this section we analyze the phenomenon of breaking of facets. From previous considerations we have shown that such objects are very stable, but, as we will see, not unbreakable. We restrict our-self to the case of operator L_1 , since we will work on the explicit formula.

Let us say few words about solving the problem

$$(L_1(u_x))_x = f$$

with the Dirichlet boundary conditions. The following scheme is a consequence of our study in [26]. We distinguish two subsets $D_0 = \{u_x = 0\}$ and $D_1 = \{u_x \neq 0\}$. On each connected component of D_1 we solve the equation $u_{xx} = f$ with $u_x = 0$ at the ends, or with the Dirichlet data if the interval touches 0 or 1. On parts for D_0 we have the energy constraint, ie. $\int_{I_1} f = 2$, where I_1 is an interval from D_0 . To justify the boundary condition $u_x = 0$ we note that the steady solution belongs to W_∞^2 , so u_x must be a continuous function, hence on the points joining components from D_0 and D_1 we obtain the constraint $u_x = 0$. Briefly, it explains solvability of the stationary problem.

Our procedure is the following. We prescribe two steady states: initial one being a construction from a constant force; and final state being a broken facet generated by a specially chosen force. First let us determine the initial state.

Lemma 6 *The solution to the problem*

$$L(u_x)_x = 4 \text{ in } (0, 1), \quad u(0) = u(1) = 0 \quad (6.1)$$

has the following form: it is a convex function, symmetric with respect of axis $\{x = 1/2\}$ such that

$$u(s) = \text{const.} \quad \text{for } s \in [1/4, 3/4]. \quad (6.2)$$

The minimum is realized at the value $-1/32$.

Proof. By the results from [26] $u \in W_\infty^2(I)$ and the minimum is realized on interval $[\xi_-, \xi_+]$

$$2 = \int_{\xi_-}^{\xi_+} L(u_x)_x dx = 4(\xi_+ - \xi_-).$$

The symmetry implies (6.2). □

The second step is to determine the final state given by a more complex force. We would like to add a force acting only on the facet, but without change of the total energy. Hence we consider the following modification

$$f_\alpha(x) = 4 + \alpha[-2\chi_{[3/8, 5/8]} + \chi_{[1/4, 3/4]}] \quad \text{with } \alpha \geq 0. \quad (6.3)$$

Then we investigate the problem

$$L(u_x)_x = f_\alpha \text{ in } (0, 1), \quad u(0) = u(1) = 0. \quad (6.4)$$

It is expected that for large α the solution will have two regions of convexity and one of concavity, hence there will be two local minima and one local maximum. Of course, all inner extrema must be realized over non-degenerate intervals. Using the approach from [26] we are able to solve the equation.

Lemma 7 *Let $\alpha > 12$, then the solution to (6.4) has the following properties:*

- (i) *all local interior minima are realized on intervals $[1/4, c]$ and $[d, 3/4]$ with $1/2 < c < 3/8$ and $5/8 < d < 3/4$;*
- (ii) *all local interior maximum is realized on the interval $[e, f]$ with $3/8 < e < 1/2 < f < 5/8$;*
- (iii) *the solution is symmetric with respect of axis $\{x = 1/2\}$.*

Proof. The symmetry of the solution is obvious. Ends of facets are obtain from the energy constraints. Then we show that indeed they fulfill the equation, the uniqueness will follows the thesis of Lemma. Assume that (i) and (ii) holds. The energy constraints on facets yield

$$\int_{1/4}^c f_\alpha dx = 2 \text{ so } (c - 1/4)(4 + \alpha) = 2, \text{ it is } c = \frac{2}{4 + \alpha} + \frac{1}{4}. \quad (6.5)$$

Provided $\alpha \geq 12$, we get $c \leq 3/8$. For the maximum, by the symmetry, we have

$$\int_e^{1/2} f_\alpha dx = -1, \text{ it is } e = \frac{1}{2} + \frac{1}{4 - \alpha}. \quad (6.6)$$

The same for $\alpha \geq 12$, we have $e \geq 3/8$. Thus, the ends of facets are determined. Now we are checking if really we are able to construct such solution. We define the solution on the interval $[0, 1/2]$. Then u shall fulfill

$$\begin{aligned} u_{xx} &= f_\alpha \quad \text{in } (0, 1/4), & u(0) &= 0, u_x(1/4) = 0; \\ u(s) &= \text{const.} & \text{for } s &\in [1/4, c]; \\ u_{xx} &= f_\alpha \quad \text{in } (c, e), & u_x(c) &= u_x(e) = 0, \\ u(s) &= \text{const.} & \text{for } s &\in [e, 1/2]. \end{aligned} \quad (6.7)$$

Observe that to solve (6.7)₃ the compatibility condition $\int_c^e f dx = 0$ must be fulfilled. From the definition

$$\int_c^e f dx = (3/8 - c)(4 + \alpha) + (e - 3/8)(4 - \alpha) = \frac{4 + \alpha}{8} - 2 + \frac{4 - \alpha}{8} + 1 = 0,$$

thus the condition is satisfied. It follows that the solution given by (6.7) exists, since

$$1/2 < c < 3/8 < e < 1/2, \quad \text{provided } \alpha > 12.$$

Hence the proof is done. Note that for $\alpha \leq 12$ there is no effects of breaking of the wall $[1/4, 3/4]$.

A simpler exercise it to check that

$$u^\alpha \leq u^{\alpha'} \text{ for } \alpha \leq \alpha'. \quad (6.8)$$

Let us look at the shape of solution on the interval $[c, e]$. Our analysis gives that $c(\alpha) \geq c(\alpha')$ as well as $e(\alpha) \leq e(\alpha')$. On the other hand the solution is a parabola over the set $[c, 3/8]$ and $[3/8, e]$ with the highest order coefficient equal to $4 + \alpha$ and $4 - \alpha$, receptively. Since at the touch point the first derivatives must be the same we conclude (6.8). \square

Breaking of facets. Having Lemmas 6 and 7 we are prepared to prove the main result of this note.

Lemma 8 *Let $\alpha > 12$ and u_0 be a steady solution to problem (6.1). We analyze the solution to the system*

$$u_t - L(u_x)_x = -f_\alpha(x, t) \text{ in } (0, 1) \times (0, \infty), \quad u(0, t) = u(1, t) = 0 \text{ with } u|_{t=0} = u_0, \quad (6.9)$$

where

$$f_\alpha(x, t) = 4 + \min\{t, \alpha\}[-2\chi_{[3/8, 5/8]} + \chi_{[1/4, 3/4]}]. \quad (6.10)$$

The solution u has the following structure

$$u(x, t) = u_0(x) \quad \text{for } t \leq 12.$$

$$u(x, t) \rightarrow u_\alpha(x) \quad \text{for } t \rightarrow \infty,$$

where u_α is a steady solution to (6.4) with force f_α .

Proof. Observe that taking the difference between (6.9) and (6.4) we get

$$(u - u_\alpha)_t - \partial_x[L(u_x) - L(u_{\alpha,x})] = 0 \text{ for } t > \alpha, \quad (6.11)$$

then testing by $u - u_\alpha$ we obtain

$$\frac{1}{2} \frac{d}{dt} \int_I (u - u_\alpha)^2 dx + \int_I [L(u_x) - L(u_{\alpha,x})](u - u_\alpha)_x dx = 0. \quad (6.12)$$

Since $L'(\cdot) \geq 2$

$$\frac{d}{dt} \int_I (u - u_\alpha)^2 dx + 2 \int_I (u - u_\alpha)_x^2 dx \leq 0 \quad (6.13)$$

which guarantees that $u \rightarrow u_\alpha$ exponentially in time in the L_2 norm.

As a consequence of Lemma 4 we observe stagnation for solutions for small t . Next, the solution, by (6.13) tends to u_α . Regularity and interpolation relations imply that $u(t)$ is close to u_α in the C^{1+a} -space with $a < 1$. Thus, the wall over $[1/4, 3/4]$ must break into three pieces (we apply here Lemma 5). □

Remark The above lemma point a way of breaking the facet, the force must be large enough to create a local energy with opposite sign to one defined on the original wall. In our case it is related to the parameter α , the largeness here is just restricted by the condition $\alpha > 12$.

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References

- [1] HW Alt, S Luckhaus, Quasilinear elliptic-parabolic differential equations, *Math. Z.*, **183**, (1983), no. 3, 311–341.
- [2] F Andreu, V Caselles, JM Mazón, S.Moll, The Dirichlet problem associated to the relativistic heat equation, *Math. Ann.*, **347**, (2010), 135–199.
- [3] SB Angenent, ME Gurtin, Anisotropic motion of a phase interface. Well-posedness of the initial value problem and qualitative properties of the interface, *J. Reine Angew. Math.* **446**, (1994), 1–47.
- [4] H Attouch, A Damlamian, Application des méthodes de convexité et monotonie à l'étude de certaines équations quasilineaires. *Proc. Roy. Soc. Edinburgh Sect. A* **79** (1977/78), no. 1-2, 107–129.
- [5] G Bellettini, V Caselles, M Novaga, The total variation flow in R^N , *J. Diff. Eqns* **184** (2002), 475–525.

- [6] G Bellettini, M Novaga, M Paolini: Facet-breaking for three-dimensional crystals evolving by mean curvature. *Interfaces Free Bound.* **1**, 39–55 (1999)
- [7] G Bellettini, M Novaga, M Paolini: Characterization of facet breaking for non-smooth mean curvature flow in the convex case. *Interfaces and Free Boundaries* **3**, 415–446 (2001)
- [8] M Bonforte, A Figalli, Total Variation Flow and Sign Fast Diffusion in one dimension, *J. Differential Equations*, **252**, (2012), 4455–4480.
- [9] E Bonnetier, E Bretin, A Chambolle, Consistency result for a non monotone scheme for anisotropic mean curvature flow. *Interfaces Free Bound.* **14** (2012), no. 1, 1–35.
- [10] R Choksi, Y van Gennip, A Oberman, Anisotropic total variation regularized L^1 approximation and denoising/deblurring of 2D bar codes. *Inverse Probl. Imaging*, **5** (2011), no. 3, 591–617.
- [11] T Fukui, Y Giga, Motion of a graph by nonsmooth weighted curvature. *World Congress of Nonlinear Analysts '92*, Vol. I–IV (Tampa, FL, 1992), 47–56, de Gruyter, Berlin, 1996.
- [12] H. Garcke, S Schaubeck, Existence of weak solutions for the Stefan problem with anisotropic Gibbs-Thomson law, *Adv. Math. Sci. Appl.* **21** (2011), 255–283
- [13] MH Giga, Y Giga, R Kobayashi, Very singular diffusion equation. *Adv. Studies Pure Math.* **31** (2001), 93–125.
- [14] Y Giga, P Górka, P Rybka: Evolution of regular bent rectangles by the driven crystalline curvature flow in the plane with a non-uniform forcing term, *Advances in Differential Equations* **18**, (2013), 201–242.
- [15] Y Giga, P Rybka: Facet bending driven by the planar crystalline curvature with a generic nonuniform forcing term, *J. Differential Equations*, **246**, (2009), 2264–2303.
- [16] Y Giga, P Rybka: Stability of facets of crystals growing from vapor, *Discrete Contin. Dyn. Syst.* **14** (2006), no. 4, 689–706.
- [17] Y Giga, P Rybka, Stability of facets of crystals growing from vapor, *Discrete Contin. Dyn. Syst.*, **14** (2006), no. 4, 689–706.
- [18] Y Giga, P Rybka, Facet bending in the driven crystalline curvature flow in the plane, *J. Geom. Anal.*, **18** (2008), no. 1, 109–147.
- [19] P Guidotti, A new nonlocal nonlinear diffusion of image processing. *J. Differential Equations* **246** (2009), no. 12, 4731–4742.
- [20] P Gwiazda, A Zatorska-Goldstein, On elliptic and parabolic systems with x -dependent multivalued graphs. *Math. Methods Appl. Sci.* **30** (2007), no. 2, 213–236.
- [21] K Kielak, PB Mucha, P Rybka, Almost classical solutions to the total variation flow, *J. Evolution Eqs*, **13**, (2013), 21–49.
- [22] R Kobayashi, Y Giga Equations with Singular Diffusivity, *Journal of Statistical Physics*, **95**, Nos. 5/6, (1999)
- [23] AM Meirmanov, The Stefan problem. Walter de Gruyter & Co., Berlin, 1992.
- [24] Y Meyer, Oscillating patterns in image processing and nonlinear evolution equations. University Lecture Series, **22**. AMS, Providence, RI, 2001.
- [25] JS Moll, The anisotropic total variation flow, *Math. Ann.*, **332** (2005), no. 1, 177–218.
- [26] PB Mucha, Regular solutions to a model monodimensional system with discontinuous elliptic operator, *Interfaces Free Bound.* **14** (2012), 145–152.
- [27] PB Mucha, P Rybka, A caricature of a singular curvature flow in the plane, *Nonlinearity*, **21**, (2008), 2281–2316.
- [28] PB Mucha, P Rybka, Well-posedness of sudden directional diffusion equations, *Math. Meth. Appl. Sci.* **36**, (2013), 2359–2370.
- [29] PB Mucha, P Rybka, A note on a model system with sudden directional diffusion, *J. Statistical Physics*, **146**, (2012), 975–988.
- [30] PB Mucha, P Rybka, Almost classical solutions of static Stefan type problems involving crystalline curvature, in: “Nonlocal and Abstract Parabolic Equations and their Applications”, Banach Center Publ. **86**, 2009, 223–234.
- [31] PB Mucha, P Rybka, A new look at equilibria in Stefan type problems in the plane, *SIAM J. Math. Anal.*, **39**, No. 4, (2007), 1120–1134.

- [32] H Spohn, Surface dynamics below the roughening transition, *J. de Physique I*, **3**, (1993), 68-81.
- [33] JE Taylor, JW Cahn, CA Handwerker, Overview no. 98 I-Geometric models of crystal growth, *Acta metall. mater.*, **40**, No. 7, (1992), 1443-1474.
- [34] JE Taylor, Overview no. 98 II-Mean curvature and weighted main curvature, *Acta metall. mater.*, **40**, No. 7, (1992) 1475-1485.
- [35] JE Taylor, Motion of curves by crystalline curvature, including triple junctions and boundary points, *Proc. Symp. Pure Math.*, **54** (1993), 417-438.